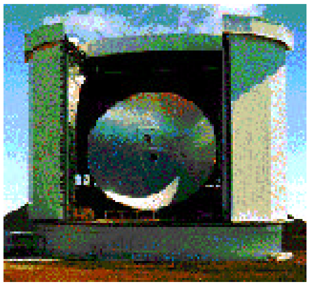

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James Clerk Maxwell Telescope
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JCMT TCS Design Note 10.0

Solving the General 4-phase routing problem

Outline of the basic routing problem.

The basic routed move required for the JCMT is a four stage move with two accelerations and two coast periods and no control over jerk (the differential of acceleration). The main difference between the JCMT and most other telescopes is that the route must finish at a pre-determined time and so the time constraint limits the flexibility of any algorithm you use. The problem can be expressed as the following five equations in 11 variables.

$$s = \frac{v_i + v_2}{2} \cdot t_1 + v_2 \cdot t_2 + \frac{v_f + v_2}{2} \cdot t_3 + v_f \cdot t_4$$

$$A_i = \frac{v_2 - v_i}{t_1} \qquad \text{i.e.} \qquad t_1 = \frac{v_2 - v_i}{A_i}$$

$$A_f = \frac{v_f - v_2}{t_3} \qquad \text{i.e.} \qquad t_3 = \frac{v_f - v_2}{A_f}$$

$$T = t_1 + t_2 + t_3 + t_4$$

$$|A_i| = |A_f|$$ i.e. the magnitudes of the accelerations are equal.

The 11 variables are:

1. s - Total distance from start to finish of the move
2. T - Total time to make the move
3. v_i - Initial velocity at the start of the move
4. A_i - Initial acceleration at start of the move
5. t_1 - Duration of the period of initial acceleration
6. v_2 - Coasting velocity after initial acceleration (i.e. the maximum velocity).
7. t_2 - Duration of the coasting velocity period.
8. A_f - Final acceleration from coasting velocity to final velocity (typically a decelleration).
9. t_3 - Duration of the period of final acceleration.
10. v_f - Final velocity
11. t_4 - Duration system is at the final velocity before the move is complete.

Given any 6 variables we can solve for the other 4. If we assume we always know v_i, v_f, s and $A_{max}=|A_i|$, this leaves 6 variables we may have to solve for. If we assume we never know t_1 and t_3 (although they are trivial if v_2 is known), then this leaves four variables left. Thus there are 4*3/2=6 cases we have to deal with. I will go through each of these cases in turn.

Case 1. Given v_i, v_f, s, A_{max}, t_2 and t_4 find v_2, T, t_1, t_3.

In this circumstance there are two regions, one in which v_2 is between v_i and v_f, and the other in which it is either smaller or greater than both. In the first case $A_i=A_f$, and in the second $A_i=-A_f$. Which region the solution lies is determined by two critical distances - the first corresponding to $v_2=v_i$ and the second when $v_2=v_f$:

$$s_{c1} = \frac{v_i + v_f}{2} \cdot \left| \frac{v_i - v_f}{A_{max}} \right| + v_i \cdot t_2 + v_f \cdot t_4$$

$$s_{c2} = \frac{v_i + v_f}{2} \cdot \left| \frac{v_i - v_f}{A_{max}} \right| + v_f \cdot t_2 + v_f \cdot t_4$$

And, if $s>\max(s_{c1},s_{c2})$ $A_i= A_{max}$ and Af=-Amax; if $s<\min(s_{c1},s_{c2})$ $A_i= -A_{max,}$ and we can solve the problem in the following way:

Substitute for t_1 & t_3:

$$s = \frac{v_i + v_2}{2} \cdot \frac{v_2 - v_i}{A_i} + v_2 \cdot t_2 + \frac{v_f + v_2}{2} \cdot \frac{v_f - v_2}{-A_i} + v_f \cdot t_4$$

Solve for v_2:

$$v_2 = \begin{bmatrix} \dfrac{-A_i \cdot t_2}{2} - \sqrt{\dfrac{t_2^{\,2} \cdot A_i^{\,2}}{4} + \dfrac{v_i^{\,2} + v_f^{\,2}}{2} + A_i \cdot (s - v_f \cdot t_4)} \\ \dfrac{-A_i \cdot t_2}{2} + \sqrt{\dfrac{t_2^{\,2} \cdot A_i^{\,2}}{4} + \dfrac{v_i^{\,2} + v_f^{\,2}}{2} + A_i \cdot (s - v_f \cdot t_4)} \end{bmatrix}$$

From which it is easy to find t_1, t_3 and T. The correct solution is the one in which v_2 is on the correct side of v_i and v_f, as determined by the critical distance test given above.

To highlight the significance of the critical distances s_c's we can consider the case where:

$$s_{c1} = \frac{v_i + v_f}{2} \cdot \frac{v_i - v_f}{A_i} + v_i \cdot t_2 + v_f \cdot t_4$$

Now let:

$$s' = s - s_{c1}$$

So

$$v_2 = \begin{bmatrix} \dfrac{-A_i \cdot t_2}{2} - \sqrt{\dfrac{t_2^{\,2} \cdot A_i^{\,2}}{4} + \dfrac{v_i^{\,2} + v_f^{\,2}}{2} + A_i \cdot (s_{c1} + s' - v_f \cdot t_4)} \\ \dfrac{-A_i \cdot t_2}{2} + \sqrt{\dfrac{t_2^{\,2} \cdot A_i^{\,2}}{4} + \dfrac{v_i^{\,2} + v_f^{\,2}}{2} + A_i \cdot (s_{c1} + s' - v_f \cdot t_4)} \end{bmatrix}$$

And substituting for s_{c1}

$$v_2 = \begin{bmatrix} \dfrac{-A_i \cdot t_2}{2} - \sqrt{\dfrac{t_2^{\,2} \cdot A_i^{\,2}}{4} + \dfrac{v_i^{\,2} + v_f^{\,2}}{2} + A_i \cdot \left(\dfrac{v_i + v_f}{2} \cdot \dfrac{v_i - v_f}{A_i} + v_i \cdot t_2 + v_f \cdot t_4 + s' - v_f \cdot t_4 \right)} \\ \dfrac{-A_i \cdot t_2}{2} + \sqrt{\dfrac{t_2^{\,2} \cdot A_i^{\,2}}{4} + \dfrac{v_i^{\,2} + v_f^{\,2}}{2} + A_i \cdot \left(\dfrac{v_i + v_f}{2} \cdot \dfrac{v_i - v_f}{A_i} + v_i \cdot t_2 + v_f \cdot t_4 + s' - v_f \cdot t_4 \right)} \end{bmatrix}$$

Which simplifies to:

$$v_2 = \begin{bmatrix} \dfrac{-A_i \cdot t_2}{2} - \sqrt{\left(\dfrac{t_2 \cdot A_i}{2} + v_i \right)^2 + s' \cdot A_i} \\ \dfrac{-A_i \cdot t_2}{2} + \sqrt{\left(\dfrac{t_2 \cdot A_i}{2} + v_i \right)^2 + s' \cdot A_i} \end{bmatrix}$$

Similarly, if we have the opposite set of signs to get a positive time in the critical distance expression, we can choose the other critical distance putting:

$$s'' = s - s_{c2}$$

Which produces the nice result:

$$v_2 = \begin{bmatrix} \dfrac{-t_2 \cdot A_i}{2} - \sqrt{\left(v_f + \dfrac{t_2 \cdot A_i}{2} \right)^2 + 4 \cdot s'' \cdot A_i} \\ \dfrac{-t_2 \cdot A_i}{2} + \sqrt{\left(v_f + \dfrac{t_2 \cdot A_i}{2} \right)^2 + 4 \cdot s'' \cdot A_i} \end{bmatrix}$$

For the other two cases Ai=Af, and Ai=Amax if vf>vi. The problem can be solved in the following way:

Substitute for t_1 & t_3: $$s = \frac{v_i + v_2}{2} \cdot \frac{v_2 - v_i}{A_i} + v_2 \cdot t_2 + \frac{v_f + v_2}{2} \cdot \frac{v_f - v_2}{A_i} + v_f \cdot t_4$$

Solve for v_2: $$v_2 = \frac{1}{t_2} \cdot \left(s + \frac{v_i^{\,2} - v_f^{\,2}}{2 \cdot A_i} - v_f \cdot t_4 \right)$$

Case 2: Given v_i, v_f, s, A_{max}, v_2 and t_4, find T, t_1, t_2 t_3.

Immediately, we have: $t_1 = \left| \dfrac{v_2 - v_i}{A_{max}} \right|$ $\qquad t_3 = \left| \dfrac{v_2 - v_f}{A_{max}} \right|$

And so we can easily find t2: $$t_2 = \frac{1}{v_2} \cdot \left(\frac{v_i + v_2}{2} \cdot t_1 + \frac{v_f + v_2}{2} \cdot t_3 + v_f \cdot t_4 \right)$$

And T is just the sum of the four component times.

Case 3: Given v_i, v_f, s, A_{max}, t_2 and T, find v_2, t_1, t_3 and t_4.

In this circumstance there are again two regions, one in which v_2 is between v_i and v_f, and the other in which it is either smaller or greater than both. In the first case A_i=A_f, and in the second A_i=-A_f. Which region the solution lies is determined by the two critical distances - the first corresponding to v_2=v_i and the second when v_2=v_f:

$$s_{c1} = \frac{v_i + v_f}{2} \cdot \left| \frac{v_i - v_f}{A_{max}} \right| + v_i \cdot t_2 + v_f \cdot \left(T - t_2 - \left| \frac{v_i - v_f}{A_{max}} \right| \right)$$

$$s_{c1} = \frac{v_i + v_f}{2} \cdot \left| \frac{v_i - v_f}{A_{max}} \right| + v_f \cdot t_2 + v_f \cdot \left(T - t_2 - \left| \frac{v_i - v_f}{A_{max}} \right| \right)$$

Which can be simplified to:

$$s_{c1} = \frac{v_i - v_f}{2} \cdot \left| \frac{v_i - v_f}{A_{max}} \right| + (v_i - v_f) \cdot t_2 + v_f \cdot T$$

$$s_{c2} = \frac{v_i - v_f}{2} \cdot \left| \frac{v_i - v_f}{A_{max}} \right| + v_f \cdot T$$

And, if s>max(s_{c1},s_{c2}) A_i= A_{max} and A_f=-A_{max}; if s<min(s_{c1},s_{c2}) A_i= -A_{max}, and we can solve the problem in the following way:

Substitute for t_4

$$s = \frac{v_i + v_2}{2} \cdot t_1 + v_2 \cdot t_2 + \frac{v_f + v_2}{2} \cdot t_3 + v_f \cdot (T - t_1 - t_2 - t_3)$$

Substitute for t_1 & t_3

$$s = \frac{v_i + v_2}{2} \cdot \frac{v_2 - v_i}{A_i} + v_2 \cdot t_2 + \frac{v_f + v_2}{2} \cdot \frac{v_f - v_2}{-A_i} + v_f \cdot \left(T - \frac{v_2 - v_i}{A_i} - t_2 - \frac{v_f - v_2}{-A_i} \right)$$

Solve for v_2:

$$v_2 = 0.5 \cdot \begin{bmatrix} 2 \cdot v_f - A_i \cdot t_2 - \sqrt{t_2^2 \cdot A_i^2 + 2 \cdot (v_i - v_f)^2 + 4 \cdot A_i \cdot (s - v_f \cdot T)} \\ 2 \cdot v_f - A_i \cdot t_2 + \sqrt{t_2^2 \cdot A_i^2 + 2 \cdot (v_i - v_f)^2 + 4 \cdot A_i \cdot (s - v_f \cdot T)} \end{bmatrix}$$

From which it is easy to find t_1, t_3 and t_4. The correct solution is the one in which v_2 is on the correct side of v_i and v_f, as determined by the critical distance test given above.

Note that once again the solution can be re-written as:

If $\frac{v_i - v_f}{A_i} > 0$

$$v_2 = \begin{bmatrix} v_f - \frac{t_2 \cdot A_i}{2} - \sqrt{\left[\left(v_f - \frac{t_2 \cdot A_i}{2} \right) - v_i \right]^2 + s' \cdot A_i} \\ v_f - \frac{t_2 \cdot A_i}{2} + \sqrt{\left[\left(v_f - \frac{t_2 \cdot A_i}{2} \right) - v_i \right]^2 + s' \cdot A_i} \end{bmatrix}$$

$$s' = s - s_{c1}$$

If $\frac{v_i - v_f}{A_i} < 0$ $\quad$ $v_2 = \begin{bmatrix} v_f - \frac{t_2 \cdot A_i}{2} - \sqrt{\left[\left(v_f - \frac{t_2 \cdot A_i}{2}\right) - v_f\right]^2 + s'' \cdot A_i} \\ v_f - \frac{t_2 \cdot A_i}{2} + \sqrt{\left[\left(v_f - \frac{t_2 \cdot A_i}{2}\right) - v_f\right]^2 + s'' \cdot A_i} \end{bmatrix}$ $\quad$ $s'' = s - s_{c2}$

For the other two cases $A_i=A_f$, and $A_i=A_{max}$ if $v_f>v_i$. The problem can be solved by solving for v_2 in the following equation:

$$s = \frac{v_i + v_2}{2} \cdot \frac{v_2 - v_i}{A_i} + v_2 \cdot t_2 + \frac{v_f + v_2}{2} \cdot \frac{v_f - v_2}{A_i} + v_f \cdot \left(T - \frac{v_2 - v_i}{A_i} - t_2 - \frac{v_f - v_2}{A_i}\right)$$

Solve for v_2: $\quad v_2 = \frac{1}{t_2} \cdot \left[s + \frac{(v_i - v_f)^2}{2 \cdot A_i} + v_f \cdot (t_2 - T)\right]$

Case 4: Given v_i, v_f, s, A_{max}, v_2 and T, find t_1, t_2, t_3 and t_4.

Immediately, we have: $\quad t_1 = \left|\frac{v_2 - v_i}{A_{max}}\right| \qquad t_3 = \left|\frac{v_2 - v_f}{A_{max}}\right|$

Substitute for t_4 $\quad s = \frac{v_i + v_2}{2} \cdot t_1 + v_2 \cdot t_2 + \frac{v_f + v_2}{2} \cdot t_3 + v_f \cdot (T - t_1 - t_2 - t_3)$

We can then immediately solve for t_2:

$$t_2 = \frac{1}{v_2 - v_f} \cdot \left[s - \frac{v_i + v_2}{2} \cdot t_1 - \frac{v_f + v_2}{2} \cdot t_3 - v_f \cdot (T - t_1 - t_3)\right]$$

And then we can easily obtain t_4

Case 5: Given v_i, v_f, s, A_{max}, t_4 and T, find v_2, t_1, t_2 and t_3.

In this circumstance there are again two regions, one in which v_2 is between v_i and v_f, and the other in which it is either smaller or greater than both. In the first case $A_i=A_f$, and in the second $A_i=-A_f$. Which region the solution lies is determined by the two critical distances - the first corresponding to $v_2=v_i$ and the second when $v_2=v_f$:

$$s_{c1} = \frac{v_i + v_f}{2} \cdot \left|\frac{v_i - v_f}{A_{max}}\right| + v_i \cdot \left(T - t_4 - \left|\frac{v_i - v_f}{A_{max}}\right|\right) + v_f \cdot t_4$$

$$s_{c2} = \frac{v_i + v_f}{2} \cdot \left|\frac{v_i - v_f}{A_{max}}\right| + v_f \cdot \left(T - t_4 - \left|\frac{v_i - v_f}{A_{max}}\right|\right) + v_f \cdot t_4$$

Which can be simplified to:

$$s_{c1} = \frac{v_f - v_i}{2} \cdot \left| \frac{v_i - v_f}{A_{max}} \right| + (v_f - v_i) \cdot t_4 + v_i \cdot T$$

$$s_{c2} = \frac{v_i - v_f}{2} \cdot \left| \frac{v_i - v_f}{A_{max}} \right| + v_f \cdot T$$

And, if $s > \max(s_{c1}, s_{c2})$ $A_i = A_{max}$ and $A_f = -A_{max}$; if $s < \min(s_{c1}, s_{c2})$ $A_i = -A_{max}$, and we can solve the problem in the following way:

Substitute for t_2: $$s = \frac{v_i + v_2}{2} \cdot t_1 + v_2 \cdot (T - t_1 - t_3 - t_4) + \frac{v_f + v_2}{2} \cdot t_3 + v_f \cdot t_4$$

Substitute for t_1 & t_3 $$s = \frac{v_i + v_2}{2} \cdot \frac{v_2 - v_i}{A_i} + v_2 \cdot \left(T - \frac{v_2 - v_i}{A_i} - \frac{v_f - v_2}{-A_i} - t_4 \right) + \frac{v_f + v_2}{2} \cdot \frac{v_f - v_2}{-A_i} + v_f \cdot t_4$$

Solve for v_2:

$$v_2 = 0.5 \cdot \begin{bmatrix} A_i \cdot (T - t_4) + v_i + v_f + \sqrt{\left[A_i \cdot (T - t_4) + v_i + v_f\right]^2 - 2 \cdot \left(v_i^2 + v_f^2\right) - 4 \cdot A_i \cdot (s - v_f \cdot t_4)} \\ A_i \cdot (T - t_4) + v_i + v_f - \sqrt{\left[A_i \cdot (T - t_4) + v_i + v_f\right]^2 - 2 \cdot \left(v_i^2 + v_f^2\right) - 4 \cdot A_i \cdot (s - v_f \cdot t_4)} \end{bmatrix}$$

From which it is easy to find t_1, t_3 and t_2. The correct solution is the one in which v_2 is on the correct side of v_i and v_f, as determined by the critical distance test given above.

Note, again the solution can be written as:

If $\frac{v_i - v_f}{A_i} > 0$ $$v_2 = \begin{bmatrix} A_i \cdot (T - t_4) + v_i + v_f - \sqrt{\left[A_i \cdot (T - t_4) + v_i + v_f - v_i\right]^2 - s' \cdot A_i} \\ A_i \cdot (T - t_4) + v_i + v_f + \sqrt{\left[A_i \cdot (T - t_4) + v_i + v_f - v_i\right]^2 - s' \cdot A_i} \end{bmatrix} \qquad s' = s - s_{c1}$$

If $\frac{v_i - v_f}{A_i} < 0$ $$v_2 = \begin{bmatrix} A_i \cdot (T - t_4) + v_i + v_f - \sqrt{\left[A_i \cdot (T - t_4) + v_i + v_f - v_f\right]^2 - s'' \cdot A_i} \\ A_i \cdot (T - t_4) + v_i + v_f + \sqrt{\left[A_i \cdot (T - t_4) + v_i + v_f - v_f\right]^2 - s'' \cdot A_i} \end{bmatrix} \qquad s'' = s - s_{c2}$$

For the other two cases $A_i = A_f$, and $A_i = A_{max}$ if $v_f > v_i$. The problem can be solved by solving for v_2 in the following equation:

$$s = \frac{v_i + v_2}{2} \cdot \frac{v_2 - v_i}{A_i} + v_2 \cdot \left(T - \frac{v_2 - v_i}{A_i} - \frac{v_f - v_2}{A_i} - t_4 \right) + \frac{v_f + v_2}{2} \cdot \frac{v_f - v_2}{A_i} + v_f \cdot t_4$$

Solve for v_2: $$v_2 = \frac{\left[s + \frac{1}{2 \cdot A_i} \cdot \left(v_i^2 - v_f^2\right) - v_f \cdot t_4 \right]}{T - \frac{v_f - v_i}{A_i} - t_4}$$

Case 6: Given v_i, v_f, s, A_{max}, v_2 and t_2, find T, t_1, t_3 and t_4.

Immediately, we have: $t_1 = \left| \frac{v_2 - v_i}{A_{max}} \right|$ $t_3 = \left| \frac{v_2 - v_f}{A_{max}} \right|$

And we can find t_4: $t_4 = \frac{1}{v_f} \cdot \left(s - \frac{v_i + v_2}{2} \cdot t_1 - v_2 \cdot t_2 - \frac{v_f + v_2}{2} \cdot t_3 \right)$

And hence T.