SCIENCE & ENGINEERING RESEARCH COUNCIL MTUN/206
JOINT ASTRONOMY CENTRE
James Clerk Maxwell Telescope
MT User Note 206

William D. Duncan
10th August 1992

Expectation Values of Signal Levels in the Presence of Refraction

Contents

1 Summary

This document describes the use of the pointing rms values determined from the CFA phase monitor to give an indication of the expectation values of the signal levels, relative to the values for no refraction 'noise'. The assumptions are

- the refraction noise can be described by a Gaussian process, which in two dimensions gives the Rayleigh distribution :

$$P_\theta = \frac{\theta}{\sigma^2} exp(\frac{-\theta^2}{2\sigma^2}) \tag{1}$$

  where P_θ is the probability of the refraction having a value of θ arcsecs, and σ is the rms refraction 'noise'.

- the telescope beam can also be defined as a Gaussian :

$$S(\theta) = S_o exp[-4ln(2)\frac{\theta^2}{\theta_{fwhm}^2}] \tag{2}$$

  where S is the signa level which would be measured for a movement of the telescope of θ arcsecs, and θ_{fwhm} is the FWHM size of the beam in arcsecs.

2 Outline of the analysis

This proceeds as in [1]. The expectation value of the signal level, relative to the no refraction noise case, is given by

$$\frac{< S >}{S_o} = \frac{1}{S_o}\int_0^{S_o} S\ P_S\ ds \tag{3}$$

where P_S is the probability distribution for the signal.
Probabilities transform directly so that

$$P_S\, dS = P_\theta(-d\theta) \tag{4}$$

where use is made of the fact that the probability of S decreasing is a function of the probability of θ increasing.
For simplicity, we rewrite (2) as

$$S(\theta) = S_o exp[-\frac{\theta^2}{2\beta^2}] \tag{5}$$

where $\beta^2 = \frac{\theta^2_{fwhm}}{8ln2}$.
Now, from equation (5),

$$\frac{dS}{d\theta} = -\frac{\theta}{\beta^2}S \tag{6}$$

using this and equations (1),(5),

$$P_S = \frac{\beta^2}{\sigma^2}\frac{1}{S}[\frac{S}{S_o}]^{\frac{\beta^2}{\theta^2}} \tag{7}$$

It should be noted that, if $\beta^2 = \sigma^2$, the probability is uniform.
If we now evaluate (3) using (7) we find that

$$\frac{<S>}{S_o} = \frac{1}{1+\frac{\sigma^2}{\beta^2}} = \frac{1}{1+\frac{8ln2\sigma^2}{\theta^2_{fwhm}}} \tag{8}$$

In the case of a perfect 15-metre telescope, one can write [4] for θ_{fwhm} (in arcsecs) :

$$\theta_{fwhm} = [14.025 + 0.1856T_E]\lambda \tag{9}$$

where T_E is the illumination edge taper in db's and λ is the wavelength in mm. In the case of a telescope with surface imperfections one must use the measured value of the full width half-maximum beam size in (8).

In order to use this result I suggest that we plot, say for beam widths at 1.1mm and 0.45mm, equation (8) with the 1 minute rms phase error from the CFA monitor. This should give some idea of the quality of the observing conditions during a run and be one more factor in assessing the quality of the data taken. It would seem that there is often not a strong correlation with the observed 225GHz tau from the CSO with the CFA phase data. A stronger correlation would seem to exist with local humidity, though this is not always a strong correlation as isolated patches of poor seeing come and go with no apparent correlation with local humidity. However, experience would suggest that whenever there is a strongly varying local humidity there will be considerable refraction 'noise'.

If we assume a 19" beam at 1.1mm and a 8" beam at 0.45mm, then (8) becomes

$$\frac{1}{1\ +\ 0.0154\sigma^2}\ \ for\ the\ 19\text{"}\ beam \tag{10}$$

and

$$\frac{1}{1\ +\ 0.0866\sigma^2}\ \ for\ the\ 8\text{"}\ beam \tag{11}$$

The analysis can be pushed a little further in order to calculate the expectation value of the signal variance and an estimate of the S/N achievable in the absence of other noise processes. The signal variance is given by

$$<S^2>\ =\ <[S-<S>]^2> \tag{12}$$

which using the results of Appendix 1, is

$$<S^2>\ =\ S_o^2\frac{a^2}{[1+2a][1+a]^2} \tag{13}$$

where a = $\frac{\sigma^2}{\beta^2}$. The signal to noise, using (18) to represent the signal level and $\sqrt{\frac{<S^2>}{bT}}$ to represent the error in the mean, is

$$S/N = \frac{1}{a}\sqrt{1+2a}\sqrt{bT} \tag{14}$$

This will overestimate the signal to noise after any given time T as there will be low frequency components to the power spectrum of the refraction which will not integrate down as $\sqrt{T}$. b is a factor to allow for the filtering effect of the integrator (i.e. data acquisition system) on the noise. Measurement of the auto-correlation function of the signal as measured through the data acquisition system, or by using Allan variance techniques, can estimate the way in which the S/N will improve with time. In general a good estimate of the signal will only be obtained after times which exceed the correlation time of the refracted noise. Even without knowledge of the noise spectrum, (14) can be used crudely to estimate the flux limit, for a given amount of refraction noise, above which a system will be dominated by refraction noise. If the current NEFD of the sytem, due to other noise processes, such as skynoise (emission noise), detector noise or photon noise is $S_{NEFD}(Jy/\sqrt{b'\,T})$, then the system is dominated by refraction noise for sources of flux greater than

$$\sim\; S_{NEFD}\frac{1}{a}\sqrt{1+2a} \tag{15}$$

Janskys. Stated conversley, this is the flux below which the system noise will dominate.

3 Further Improvements in the use of the CFA phase monitor

- The rms phase error reported by the ”PHASE” command is a power law extrapolation from the 100m baseline of the 12GHz phase monitor to the 15m baseline of the JCMT. The conversion factor from the CFA data, (in rms degrees (determined over 1 minute if integration) of phase at 11.8554GHz for a 100m baseline), to an estimate of the rms arcsecs on a 15m baseline (JCMT), is given by [2]

  $$\sigma(arcsecs\,rms) \sim \sigma_{CFA}(deg\,rms)\;*\;0.5 \tag{16}$$

  where a scaling of path difference $\propto (baseline)^{0.75}$ is used to scale between 100 and 15 metres. A further factor of 2 is embedded in (16) becasue the noise process is non-white and hence the 1 minute estimates are correlated and tend to underestimate the true rms by a factor of $\sim$2. Under some conditions this factor may be larger. The extrapolation is also only valid in the atmospheric windows (i.e. away from strong absorption lines) and it is not always possible to avoid these. As the extrapolation of (16) is somewhat uncertain, a series of measurements to confirm, or otherwise, the scaling factor of (16) should be made. Some direct measurements of seeing as these wavelengths have already been done by Church and Hills [3].

- Further work on correlating the CFA measurements with the CSO tau and local humidity would be very useful in characterising the site and understanding the limitations of the atmospheric monitoring equipment that are currently deployed on the mountain.

4 Appendix 1

Useful moments of the probability function are given below :

$$\int_0^{S_o} P_S dS = 1 \tag{17}$$

$$\int_0^{S_o} S\,P_S dS = S_o\frac{1}{1+\frac{\sigma^2}{\beta^2}} \tag{18}$$

$$\int_0^{S_o} S^2 \, P_S dS = S_o^2 \frac{1}{1 + 2\frac{\sigma^2}{\beta^2}} \tag{19}$$

The integrals are, perhaps, best evaluated with a simple change of variable - eg let $x = S/S_o$, and integrating between 0 and 1.

References

[1] D.L.Fried, Applied Optics, Vol 12, No. 2, 1973, pp 422 - 423

[2] C.Masson, private communication, 1991.

[3] S.E.Church, R.E.Hills, 1990, *Proc. URSI/IAU symposium* 'Radio Astronomical Seeing', eds J.E.Baldwin and Wang Shouguan, Beijing.

[4] P.F.Goldsmith, International Journal of Infrared and Millimeter Waves, Vol 8, No. 7, pp 771 - 781, 1987.